

A NOTE ON (ANTI-)SELF DUAL QUASI YAMABE SOLITON

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ABSTRACT. In this note we prove that a (anti-)self dual quasi Yamabe soliton with positive sectional curvature is rotationally symmetric. This generalizes a recent result of G. Huang and H. Li in dimension four. Whence, (anti-) self dual gradient Yamabe solitons with positive sectional curvature is rotationally symmetric. We also prove that half conformally flat gradient Yamabe soliton has a special warped product structure provided that the potential function has no critical point.

1. INTRODUCTION

The Yamabe flow was introduced by R. Hamilton in an attempt to prove the Yamabe problem (see [10]). Solitons are important to understand the geometric flow since they can appear as singularity models. Yamabe solitons are self-similar solutions for the Yamabe flow.

Definition 1. A Riemannian manifold (M^n, g) with dimension $n \geq 3$ is called a gradient Yamabe soliton if there exist a smooth potential function $f : M^n \rightarrow \mathbb{R}$ and a constant λ such that

$$(1.1) \quad (R - \lambda)g_{ij} = \nabla_i \nabla_j f.$$

If $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, then we have, respectively, a gradient Yamabe soliton shrinker, steady and expanding.

Motivated by the results on quasi Einstein manifolds (see [6]), the theory of quasi Yamabe gradient solitons started to be investigated. Quasi Yamabe gradient solitons are generalizations of gradient Yamabe solitons.

Definition 2. A quasi Yamabe gradient soliton is a triple (M^n, g, f) , where (M^n, g) is a Riemannian manifold of dimension $n \geq 3$ with a smooth potential function $f : M^n \rightarrow \mathbb{R}$ and two constants λ, m ($m \neq 0$) satisfying

$$(1.2) \quad (R - \lambda)g_{ij} = \nabla_i \nabla_j f - \frac{1}{m} \nabla_i f \nabla_j f.$$

When $m \rightarrow \infty$ then (1.2) reduces to (1.1) and if f is constant we say that the quasi Yamabe gradient soliton is trivial.

Recently 4-dimensional manifolds has been widely studied (see [1], [2], [6], [7] and [11]). In what follows M^4 will denote an oriented 4-dimensional manifold and g is a Riemannian metric on M^4 . We emphasize that 4-manifolds are fairly special. For instance, following the notations used in [9] (see also [12] and [3]), given any local orthogonal frame $\{e_1, e_2, e_3, e_4\}$ on open set of M^4 with dual basis $\{e^1, e^2, e^3, e^4\}$, there exists a unique bundle morphism $*$ called *Hodge star* (acting on bivectors), such that

$$*(e^1 \wedge e^2) = e^3 \wedge e^4.$$

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This implies that $*$ is an involution, i.e. $*^2 = Id$. In particular, this implies that the bundle of 2-forms on a 4-dimensional oriented Riemannian manifold can be invariantly decomposed as a direct sum $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$. This allows us to conclude that the Weyl tensor W is an endomorphism of $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ such that

$$(1.3) \quad W = W^+ \oplus W^-.$$

We recall that $\dim_{\mathbb{R}}(\Lambda^2) = 6$ and $\dim_{\mathbb{R}}(\Lambda^{\pm}) = 3$. Also, it is well-known that

$$(1.4) \quad \Lambda^+ = \text{span} \left\{ \frac{e^1 \wedge e^2 + e^3 \wedge e^4}{\sqrt{2}}, \frac{e^1 \wedge e^3 + e^4 \wedge e^2}{\sqrt{2}}, \frac{e^3 \wedge e^2 + e^4 \wedge e^1}{\sqrt{2}} \right\}$$

and

$$(1.5) \quad \Lambda^- = \text{span} \left\{ \frac{e^1 \wedge e^2 - e^3 \wedge e^4}{\sqrt{2}}, \frac{e^1 \wedge e^3 - e^4 \wedge e^2}{\sqrt{2}}, \frac{e^3 \wedge e^2 - e^4 \wedge e^1}{\sqrt{2}} \right\}.$$

From this, the bundles Λ^+ and Λ^- carry natural orientations such that the bases (1.4) and (1.5) are positive-oriented.

The decomposition of the Weyl tensor on 4-dimensional manifolds allow us to deduce the following equation

$$(1.6) \quad W_{pqrs}^+ = \frac{1}{2}(W_{pqrs} + W_{\overline{p}\overline{q}rs}),$$

where $(\overline{p}\overline{q})$, for instance, stands for the dual of (pq) , that is, $(\overline{p}\overline{q}pq) = \sigma(1234)$ for some even permutation σ in the set $\{1, 2, 3, 4\}$ (cf. Equation 6.17, p. 466 in [9]). In particular, we have

$$W_{1234}^+ = \frac{1}{2}(W_{1234} + W_{1212}).$$

For more details we refer to [9], [12] and [3]. We say that a 4-dimensional manifold is half conformally flat if it is self or anti-self dual, namely if $W^- = 0$ or $W^+ = 0$.

Inspired by [7] and [11], we consider 4-dimensional quasi Yamabe gradient solitons. More precisely, we prove:

Theorem 1. *Let (M^4, g, f) be a nontrivial complete anti-self dual (or self dual) quasi Yamabe gradient soliton satisfying (1.2) with positive sectional curvature. Then (M^4, g, f) is rotationally symmetric.*

We have the following result from Theorem 1 immediately:

Corollary 1. *Let (M^4, g, f) be a nontrivial complete anti-self dual (or self dual) gradient Yamabe soliton satisfying (1.1) with positive sectional curvature. Then (M^4, g, f) is rotationally symmetric.*

In fact, we can improve corollary 1.

Theorem 2. *Let (M^4, g, f) be a nontrivial complete anti-self dual (or self dual) gradient Yamabe soliton satisfying (1.1) and suppose that f has no critical points. Then (M^4, g, f) is the warped product*

$$(\mathbb{R}, dr^2) \times_{|\nabla f|} (N^3, \bar{g}),$$

where (N^3, \bar{g}) is a space form (i.e, of constant sectional curvature).

Remark 1. In [8], Daskalopoulos and Sesum proved that gradient Yamabe solitons with positive sectional curvature are rotationally symmetric under the assumption that the metric g is locally conformally flat. Cao, Sun and Zhang [4] proved that gradient Yamabe solitons has a special warped product structure.

2. PRELIMINARES

The Weyl tensor W is defined by the following decomposition formula

$$(2.1) \quad \begin{aligned} R_{ijkl} &= W_{ijkl} + \frac{1}{n-2}(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}) \\ &- \frac{R}{(n-1)(n-2)}(g_{jl}g_{ik} - g_{il}g_{jk}). \end{aligned}$$

It is well know that $W = 0$ for $n = 3$. A 4-dimensional manifold is locally conformally flat if and only if $W = 0$.

In [5], Cao and Chen defined the tensor D . This tensor is the link between the Weyl tensor and quasi Yamabe solitons (see proposition 1). We define the 3-tensor D by

$$(2.2) \quad \begin{aligned} D_{ijk} &= \frac{1}{n-2}(R_{jk}\nabla_i f - R_{ik}\nabla_j f) + \frac{1}{(n-1)(n-2)}(R_{il}\nabla^l f g_{jk} - R_{jl}\nabla^l f g_{ik}) \\ &- \frac{R}{(n-1)(n-2)}(g_{jk}\nabla_i f - g_{ik}\nabla_j f). \end{aligned}$$

The tensor D is skew-symmetric in their first two indices and trace-free, i.e.,

$$D_{ijk} = -D_{jik} \quad \text{and} \quad g^{ij}D_{ijk} = g^{ik}D_{ijk} = g^{jk}D_{ijk} = 0.$$

Proposition 1. [11] *Let (M^n, g, f) be a nontrivial complete quasi Yamabe gradient soliton satisfying (1.2). Then the 3-tensor D is related to the Weyl curvature tensor by*

$$D_{ijk} = W_{ijkl}\nabla^l f.$$

Theorem 3. [11] *Let (M^4, g, f) be a nontrivial complete quasi Yamabe gradient soliton satisfying (1.2) with positive sectional curvature and $D_{ijk} = 0$. Then (M^4, g, f) is rotationally symmetric.*

Theorem 4. [4] *Let (M^n, g, f) be a nontrivial complete gradient Yamabe soliton satisfying equation (1.1). Then $|\nabla f|^2$ is constant on regular level surfaces of f , and either*

(i) *f has a unique critical point at some point $x_0 \in M^n$, and (M^n, g, f) is rotationally symmetric and equal to the warped product*

$$([0, +\infty), dr^2) \times_{|\nabla f|} (\mathbb{S}^{n-1}, \bar{g}_{can}),$$

where \bar{g}_{can} is the metric on \mathbb{S}^{n-1} , or

(ii) *f has no critical point and (M^n, g, f) is the warped product*

$$(2.3) \quad (\mathbb{R}, dr^2) \times_{|\nabla f|} (N^{n-1}, \bar{g}),$$

where (N^{n-1}, \bar{g}) is a Riemannian manifold of constant scalar curvature.

When (M^n, g, f) is locally conformally flat we have

Theorem 5. [4] *Let (M^n, g, f) be a nontrivial complete gradient Yamabe soliton satisfying equation (1.1). Suppose f has no critical point and is locally conformally flat. Then (M^n, g, f) is the warped product*

$$(\mathbb{R}, dr^2) \times_{|\nabla f|} (N^{n-1}, \bar{g}),$$

where (N^{n-1}, \bar{g}) is a space form (i.e., of constant sectional curvature).

In this paper we show that the hypothesis locally conformally flat in theorem 5 can be replaced by the weaker condition (anti-)self dual Weyl tensor (or half conformally flat) on four dimensional case.

Remark 2. *It is worth point out that compact quasi Yamabe gradient solitons and compact Yamabe gradient solitons has constant scalar curvature (see [8] and [11]).*

3. PROOF OF THE MAIN RESULTS

3.1. Proof of Theorem 1.

Proof. Now, we follow the same ideas from [7]. Consider a 4-dimensional Riemannian manifold satisfying definition 2 with $W^+ = 0$, i.e, a complete anti-self dual quasi Yamabe gradient soliton. Then, from (1.6) we have

$$W_{i\bar{j}kl} + W_{i\bar{j}kl} = 0.$$

Whence,

$$W_{i\bar{j}kl}\nabla^l f + W_{i\bar{j}kl}\nabla^l f = 0.$$

This implies from proposition 1 that

$$D_{i\bar{j}k} + D_{i\bar{j}k} = 0.$$

Now, for some oriented orthonormal basis $\{e_i\}_{i=1}^4$ diagonalizing the Ricci tensor with associated eigenvalues μ_k , $k = 1, \dots, 4$, respectively, i.e, $R_{ij} = \mu_i \delta_{ij}$, we have

$$(3.1) \quad D_{12k} + D_{34k} = 0, \quad D_{13k} + D_{42k} = 0 \quad \text{and} \quad D_{14k} + D_{23k} = 0.$$

From (2.2) we get

$$(3.2) \quad D_{ijk} = 0, \quad \text{for } i \neq j \neq k.$$

Therefore, from (3.1) and (3.2) we obtain

$$(3.3) \quad D_{iji} = 0, \quad \text{for } i, j = 1, \dots, 4.$$

We also have that the tensor D is skew-symmetric, i.e, $D_{iij} = 0$. Finally we get that $D \equiv 0$. Since the sectional curvature is positive we can apply theorem 3 to get the result. \square

3.2. Proof of Theorem 2.

Proof. Let (M^4, g, f) be a complete anti-self dual (or half conformally flat) gradient Yamabe soliton such that f has no critical point. From theorem 4-(ii) we get that (M^4, g, f) has a warped product structure (2.3). We want to prove that (N^3, \bar{g}) is a space form. For this, it suffices to show that \bar{g} is Einstein. Since three dimensional Einstein manifolds has constant sectional curvature. To conclude our result we need to know the warped product structure of complete gradient Yamabe soliton (see [4]).

By theorem 1 and proposition 1 we already know that $W_{ijkl}\nabla^l f = 0$, i.e,

$$(3.4) \quad W(\cdot, \cdot, \cdot, \nabla f) = 0.$$

Consider the level surface $\Sigma = f^{-1}(c)$ where c is any regular value of the potential function f . Suppose that I is an open interval containing c such that f has no critical point in the open neighborhood $U_I = f^{-1}(I)$. Since $|\nabla f|^2$ is constant on Σ , we can make a change of variable so that we can express the metric g in U_I as in theorem 4-(ii) (see [4]). Fix any local coordinates system $\theta = (\theta_2, \theta_3, \theta_4)$ on N^3 , and choose $(x_1, x_2, x_3, x_4) = (r, \theta_2, \theta_3, \theta_4)$. Let $\nabla r = \frac{\partial}{\partial r}$, then $|\nabla r| = 1$ and $\nabla f = f'(r)\frac{\partial}{\partial r}$ on U_I . Whence, we get from (3.4)

$$0 = W(\cdot, \cdot, \cdot, \nabla f) = f'(r)W(\cdot, \cdot, \cdot, \nabla r).$$

Since f has no critical point we get

$$(3.5) \quad W(\cdot, \cdot, \cdot, \nabla r) = 0.$$

Therefore, from arguments on [4] (see theorem 1.4), the Weyl tensor formula for an arbitrary warped product dimensional manifold give us

$$(3.6) \quad 2W(\nabla r, \theta_a, \nabla r, \theta_b) = \frac{\bar{R}}{3}\bar{g}(\theta_a, \theta_b) - \bar{Ric}(\theta_a, \theta_b), \quad \text{for } a, b \in \{2, 3, 4\},$$

where \bar{R} and \bar{Ric} stand, respectively, for the scalar curvature and Ricci tensor for (N^3, \bar{g}) . Therefore, from (3.5) and (3.6) we have theorem 2. \square

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